$X \sim N(\mu, \sigma^2)$

We draw a random sample of size n $x_1, x_2, \ldots, x_n$ from $N(\mu, \sigma^2)$, σ^2 is known. [Only unknown parameter is μ]

I) Test $H_0: \mu = \mu_0$ under H_0, $\mu = \mu_0$ (a proposed value)

$H_1: \mu > \mu_0$

Test statistic $\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$

$\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$

$\therefore Z = \frac{\sqrt{n}(\bar{x} - \mu)}{\sigma} \sim N(0,1)$

For H_0 is true,

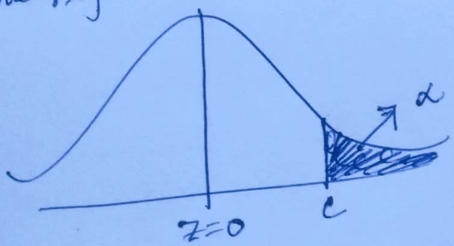
$Z = Z_{calculated}$

$= \frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma} \sim N(0,1)$

Note

Even if $x_1, x_2, \ldots, x_n$ are not from Normal, for large sample, say $n > 10$, $\frac{\Sigma x_i - E(\Sigma x_i)}{\sqrt{V(\Sigma x_i)}} \to N(0,1)$ by central limit theorem

For testing $H_1: \mu > \mu_0$, the rejection region will be on the right side of the bell shaped normal curve as (right tail) the larger the value of μ from μ_0, H_0 is going to be false hence rejected.



Fix level of significance as α and c being the cut off (critical point) beyond which we reject H_0.

The value of c will be calculated from the size condition, i.e.,

$\alpha = P_{H_0}[Z_{calculated} > c]$

$\alpha = P_{H_0}\left[\frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma} > c\right]$

$\therefore c = Z_\alpha$ where Z standard normal and α being right tail probability | area

when $\alpha = .05\ (5\%)$; $Z_\alpha = 1.64$

when $\alpha = .01\ (1\%)$; $Z_\alpha = 2.33$

Therefore we reject H_0 if.

$$z_{calculated} = \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} > Z_\alpha \text{ (c)}$$

$$\Rightarrow \boxed{\bar{x} > \mu_0 + \frac{\sigma}{\sqrt{n}} \cdot Z_\alpha} \rightarrow \text{rejection rule.}$$

$$\boxed{\bar{x} < \mu_0 + \frac{\sigma}{\sqrt{n}} Z_\alpha} \rightarrow \text{acceptance / fail of rejection rule.}$$

In particular, say $H_0: \mu = \mu_0 = 5$
$H_1: \mu > 5$, $\alpha = .05$, $\sigma = 2$ and $n = 10$

then what is rejection rule?

$$\bar{x} > \mu_0 + \frac{\sigma}{\sqrt{n}} \cdot Z_\alpha = 5 + \frac{2}{\sqrt{10}} \cdot 1.64 = 5 + 1.037$$
$$= 6.037$$

Then if the sample collected $x_1, x_2, \cdots, x_n$ results the $\bar{x} = 7$ (say) > 6.037, we reject H_0 on the basis of the sample.

And if the sample mean $\bar{x} = 2$ (say) $2 < 6.037$, we accept H_0.

Power: $H_1: \mu > \mu_0$.

power $= Pr$ [reject a false null hypo]
$= Pr$ [rejection rule / H_1 is true]
$$= Pr\left[\bar{x} > \mu_0 + \frac{\sigma}{\sqrt{n}} \cdot Z_\alpha \,/\, H_1 \text{ is true}\right]$$
$$= Pr\left[\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma} > \frac{(\mu_0 + \frac{\sigma}{\sqrt{n}} Z_\alpha - \mu)\sqrt{n}}{\sigma}\right]$$
$$= Pr\left[Z > Z_\alpha - \frac{(\mu-\mu_0)\sqrt{n}}{\sigma}\right]$$
$$= \Phi\left(Z_\alpha - \frac{\sqrt{n}(\mu-\mu_0)}{\sigma}\right)$$

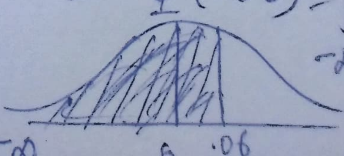
Now find what is the distribution of $\bar{x}$ under H_1?

$\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$.

particular case

For the previous problem $H_0: \mu = 5$, $H_1: \mu > 5$, $z_{.05} = 1.64$

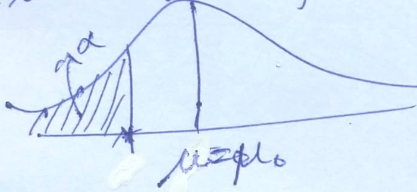
~~Pr~~	Power
$\mu = 6$	$\Phi\left(1.64 - \frac{\sqrt{10}(6-5)}{2}\right) = \Phi(1.64 - 1.58) = \Phi(.06) = N(0,1)$
	$\Phi(.06) = .5239$
$\mu = 7$	$\Phi(1.64 - \frac{\sqrt{10}(7-5)}{2})$

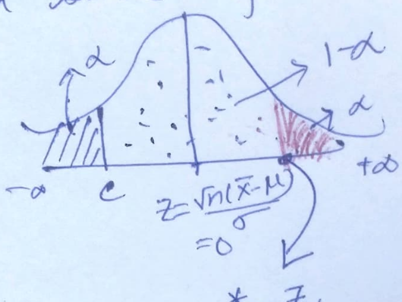


II) $H_1: \mu < \mu_0$

The basic change for this left tail test is smaller ~~the~~ the value of μ from μ_0. H_0 is going to be false, thereby we reject null hyp.

So, rejection will be on left-hand side of Gaussian curve





[Think on c and c^*]

Under H_0,

So, $\hat{Z} = Z_{calculated}$

$$= \frac{\sqrt{n}(\bar{X}-\mu_0)}{\sigma} \sim N(0,1)$$

We reject H_0 if $Z_{calculated} < c$

where c being the critical point (cut off point)

$$\alpha = P_{H_0}\left[\frac{\sqrt{n}(\bar{X}-\mu_0)}{\sigma} < c\right]$$

$$\Rightarrow 1-\alpha = P_{H_0}\left[\frac{\sqrt{n}(\bar{X}-\mu_0)}{\sigma} > c\right]$$ [why this is necessary?]

$$\Rightarrow c = Z_{1-\alpha} = -Z_\alpha$$ (because of symmetry)

Thus the rejection rule is $Z_{calculated} < Z_{1-\alpha}$

$$\Rightarrow \frac{\sqrt{n}(\bar{X}-\mu_0)}{\sigma} < Z_{1-\alpha}$$

$$\Rightarrow \bar{X} < \mu_0 + \frac{\sigma}{\sqrt{n}} Z_{1-\alpha}$$

$$\Rightarrow \boxed{\bar{X} < \mu_0 - \frac{\sigma}{\sqrt{n}} Z_\alpha}$$

We accept H_0 if $\bar{X} > \mu_0 - \frac{\sigma}{\sqrt{n}} Z_\alpha$.

Power

$$\text{power} = \Pr[\text{reject a false null hyp}]$$

$$= \Pr\left[\bar{X} < \mu_0 - \frac{\sigma}{\sqrt{n}} Z_\alpha \,\middle/\, H_1 \text{ true}\right]$$

$$= \Pr\left[\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} < \frac{\sqrt{n}\left(\mu_0 - \frac{\sigma}{\sqrt{n}} Z_\alpha - \mu\right)}{\sigma}\right]$$

$$= \Pr\left[Z < \frac{\sqrt{n}(\mu_0-\mu)}{\sigma} - Z_\alpha\right]$$

$$= \Phi\left(\frac{\sqrt{n}(\mu_0-\mu)}{\sigma} - Z_\alpha\right)$$

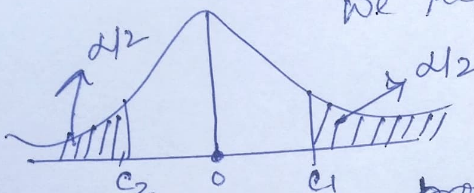
Case III $H_1: \mu \neq \mu_0$.

Rejection region will be two tail as too big μ as compared with μ_0 will as well as too small μ as compared with μ_0 will lead to reject H_0.

test statistic : $\dfrac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$

Test rule will be :

we reject H_0, if $\dfrac{\sqrt{n}(\bar{X}-\mu_0)}{\sigma} > c_1$ or $\dfrac{\sqrt{n}(\bar{X}-\mu_0)}{\sigma} < c_2$



If we fix level of significance as α, then right tail rejection probability is $\frac{\alpha}{2}$ and left tail rejection is $\frac{\alpha}{2}$ [this type of distribution is called equal tail test]. c_1 and c_2 will be calculated from size condition.

Therefore $Pr_{H_0}\left[\dfrac{\sqrt{n}(\bar{X}-\mu_0)}{\sigma} > c_1\right] = \alpha/2$ —— ①

and $Pr_{H_0}\left[\dfrac{\sqrt{n}(\bar{X}-\mu_0)}{\sigma} < c_2\right] = \alpha/2$ —— ②

$\Rightarrow$ ① $\Rightarrow c_1 = Z_{\alpha/2}$

② $\Rightarrow c_2 = Z_{1-\alpha/2} = -Z_{\alpha/2}$.

Now see ① is same as condition for $H_1: \mu > \mu_0$

② is same as condition for $H_1: \mu < \mu_0$

We use the techniques used earlier.

So we reject H_0 if

$Z_{calcu} = \dfrac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} > Z_{\alpha/2}$ or

$Z_{calcu} = \dfrac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} < -Z_{\alpha/2}$.

$\therefore$ Clubbing together

$|Z| > Z_{\alpha/2}$

$\Rightarrow \left|\dfrac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}\right| > Z_{\alpha/2}$

For $\alpha = .05$

$Z_{\alpha/2} = Z_{.025} = 1.96$

Power

$Pr\left[\left|\dfrac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}\right| > Z_{\alpha/2} \Big/ H_1\right]$

~~$= Pr\left[Z_{\alpha/2} < \dfrac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}\right]$~~

$= Pr\left[\dfrac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} > Z_{\alpha/2} \Big/ H_1\right] + Pr\left[\dfrac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} < -Z_{\alpha/2} \Big/ H_1\right]$